

Comparison of Nonrelativistic limit of Amelino-Camelia and MS Doubly Special Relativity

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Abstract

This paper is devoted to the study of the nonrelativistic limit of Amelino-Camelia Doubly Special Relativity, and the corresponding modified Klein-Gordon and Dirac equations. We show that these equations reduce to the Schrödinger equations for the particle and the antiparticle with different inertial masses. However, their rest masses are the same. M. Coraddu and S. Mignemi have studied recently the non relativistic limit of the Magueijo-Smolín Doubly Special Relativity. We compare their results with our study, and show that these two models are reciprocal to each other in the nonrelativistic limit. The different inertial masses also leads to the CPT violation.

Keywords: Doubly Special Relativity, Non-relativistic limit, Relativistic Quantum Mechanics.

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1 Introduction

Doubly Special Relativity (DSR) theories have been proposed ten years ago for the nonlinear modification of Special Relativity. These theories have two invariant scales, the speed of light c and the Planck energy $E_p = \sqrt{\hbar c^5/G} \simeq 10^{19}$ GeV. Magueijo-Smolín (MS) DSR [1] and Amelino-Camelia DSR [2, 3] are the two main examples of these theories. Although, these models have similar structure, they belong to different realizations of kappa-Poincaré algebras [4].

According to the special relativity and relativistic quantum mechanics the Schrödinger equation is the nonrelativistic limit of the Klein-Gordon and Dirac equations [5]. Also, we have the same mass for the particle and the antiparticle, and this fact can be seen easily from the dispersion relation $E = \pm\sqrt{m^2 + p^2}$.

The dispersion relation has been modified in the DSR theories, which leads to the modified Klein-Gordon and Dirac equations. It will be interesting to study the nonrelativistic limit of these modified equations. The nonlinearity of the dispersion relation and the fact that the dispersion relation is not invariant under space inversion and time reversal also imply the violation of CPT invariance.

Recently, M. Corradu and S. Mignemi have studied the nonrelativistic limit of the MS DSR and the corresponding modified Klein-Gordon and Dirac equations [6]. They illustrated that the particle and the antiparticle rest masses are different. However, their inertial masses are the same. Besides, they have showed that the modified Klein-Gordon and Dirac Equations in the MS model reproduce nonrelativistic quantum mechanics. To continue their proposal we want to study the nonrelativistic limit of Amelino-Camelia Doubly Special Relativity and the corresponding modified Klein-Gordon and Dirac equations.

We show that the corresponding modified Klein-Gordon and Dirac equations reduce the Schrödinger equations for the particle and the antiparticle with different inertial masses. The difference between these two masses is proportional to mc^2/E_p in the first order of approximation. However, their rest masses are the same. Different inertial masses also leads to the violation of CPT invariance.

We compare the nonrelativistic limit of MS and Amelino-Camelia DSR. Our results are reciprocal to the M. Corradu and S. Mignemi results. We can

interpret MS DSR as modifying rest mass [6], and Amelino-Camelia DSR as modifying momentum.

In the following section, we summarize some basics of Amelino-Camelia DSR. In section 3, we study the non-relativistic limit of modified dispersion relation in Amelino-Camelia DSR, and make some comparisons with the nonrelativistic limit of MS DSR. The non-relativistic limits of ordinary and modified Klein-Gordon and Dirac equations is presented in section 4. Then, in section 5 we have some conclusions.

2 Amelino-Camelia DSR

The essence of Amelino-Camelia DSR is the modification of ordinary special relativistic boosts

$$K_a = ip_a \frac{\partial}{\partial E} + iE \frac{\partial}{\partial p_a},$$

to preserve the following nonlinear dispersion relation invariant [3]. This modified dispersion relation is

$$2k^2 \left[\cosh\left(\frac{E}{k}\right) - \cosh\left(\frac{m}{k}\right) \right] = \mathbf{p}^2 e^{E/k}, \quad (1)$$

in which k is the Planck energy E_p and a is an index varies from 1 to 3. He proposed the modified boosts

$$B_a = ip_a \frac{\partial}{\partial E} + i \left(\frac{1}{2k} \mathbf{p}^2 + k \frac{1 - e^{-2E/k}}{2} \right) \frac{\partial}{\partial p_a} - i \frac{p_a}{k} \left(p_b \frac{\partial}{\partial p_b} \right), \quad (2)$$

which leave invariant the mentioned dispersion relation. Also, we can find new transformations for this doubly special relativity which is different from the Lorentz transformations [7].

The modified Klein-Gordon equation is obtained from (1) by substituting differential operators for $E = i \frac{\partial}{\partial t}$ and $\mathbf{p} = -i \vec{\nabla}$ in the fashion standard in quantum mechanics

$$\left[\nabla^2 - 2k^2 \exp\left(\frac{-i}{k} \frac{\partial}{\partial t}\right) \left(\cosh\left(\frac{-i}{k} \frac{\partial}{\partial t}\right) - \cosh\left(\frac{m}{k}\right) \right) \right] \Psi(\vec{x}, t) = 0. \quad (3)$$

Also, we can construct deformed Dirac equation as

$$(\gamma^\mu D_\mu - I) \Psi(\mathbf{p}) = 0 \quad (4)$$

where D_μ is the modified dirac operator

$$D_0 = \frac{e^{E/k} - \cosh(m/k)}{\sinh(m/k)},$$

$$D_a = \frac{p_a}{p} \frac{\left(2e^{E/k} [\cosh(E/k) - \cosh(m/k)]\right)^{1/2}}{\sinh(m/k)},$$

and γ^μ are the familiar Dirac γ matrices [3].

3 Nonrelativistic limit

In the ordinary special relativity, we go to the classical non-relativistic limit for a free particle by expanding ordinary dispersion relation

$$E^2 - c^2 \mathbf{p}^2 = m^2 c^4 \quad (5)$$

in $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ limit, which leads to

$$E = \pm \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \simeq \pm (mc^2 + \frac{\mathbf{p}^2}{2m} + \dots). \quad (6)$$

The first term on the right hand side is the rest mass energy and the second term is the classical kinetic energy.

In Amelino-Camelia DSR, we expand modified dispersion relation (1) in $O(E^3/k^3)$ approximation and we obtain

$$E^2 - \mathbf{p}^2 c^2 - \frac{\mathbf{p}^2 c^2}{k} E = m^2 c^4. \quad (7)$$

Solving this equation as a second order equation for E , we have

$$E = \frac{\mathbf{p}^2 c^2 / k \pm \sqrt{\mathbf{p}^4 c^4 / k^2 + 4(\mathbf{p}^2 c^2 + m^2 c^4)}}{2}.$$

In the $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ limit we have

$$E \simeq \frac{\mathbf{p}^2 c^2}{2k} \pm mc^2 \left(1 + \frac{\mathbf{p}^2}{2m^2 c^2}\right).$$

The positive sign corresponds to the classical non-relativistic limit

$$E = mc^2 + \frac{\mathbf{p}^2}{2m^+}, \quad (8)$$

in which we assume

$$m^+ = \frac{m}{1 + \frac{mc^2}{k}}. \quad (9)$$

Thus, we find that the particle inertial mass has been modified by amount of mc^2/k , but the rest mass of particle remains unmodified. By inertial mass we mean the mass which appears in dominator of $\mathbf{p}^2/2m^+$. Also, by defining

$$m^- = \frac{m}{1 - \frac{mc^2}{k}} \quad (10)$$

we can interpret the negative sign solution as an antiparticle

$$E = -mc^2 - \frac{\mathbf{p}^2}{2m^-}, \quad (11)$$

moving in the opposite direction of time with modified inertial mass m^- .

Different inertial mass m^+ and m^- in the equations (9) and (10) leads to the violation of CPT invariance. However, the interpretation of the inertial mass as the mass of an elementary particle may have some difficulties. In principle, modification of mass-shell relation Eq.(7) can be interpreted as the violation of CPT.

We can compare this case with nonrelativistic limit of the MS model [6]. They reached to the modified relation ($E = m^+c^2 + \frac{\mathbf{p}^2}{2m}$) instead of Eq.(8) in which m^+ has the same value with Eq.(9). In this model the rest mass of the particle is modified, and the inertial mass remains the same. Thus we have an interesting result, that the nonrelativistic limit of Magueijo-Smolín DSR is reciprocal to the nonrelativistic limit of Amelino-Camelia DSR.

Also, we can calculate the non-relativistic limit of the velocity of a particle. The group velocity is

$$\mathbf{v}_g = \frac{\partial E}{\partial \mathbf{p}} \simeq \frac{\mathbf{p}}{m^+}, \quad (12)$$

and the particle velocity is

$$\mathbf{v}_{particle} = \frac{\mathbf{p}c^2}{E} \simeq \frac{\mathbf{p}}{m}. \quad (13)$$

In the ordinary special relativity we have the same value of \mathbf{p}/m for the group velocity \mathbf{v}_g and the particle velocity $\mathbf{v}_{particle}$. Here, we showed that the group velocity differ from the special relativistic value but the particle velocity is the same as in the special relativity. In the nonrelativistic limit of MS DSR the situation for velocities are inverse to our case and we have $\mathbf{v}_g \simeq \mathbf{p}/m$ and $\mathbf{v}_{particle} \simeq \mathbf{p}/m^+$ [6].

We summarize comparison of the nonrelativistic limit of Amelino-Camelia and MS doubly special relativity in the following table.

Amelino-Camelia DSR	Magueijo-Smolín DSR
$E = mc^2 + \mathbf{p}^2/2m^+$	$E = m^+c^2 + \mathbf{p}^2/2m$
$\mathbf{v}_g = \mathbf{p}/m^+$	$\mathbf{v}_g = \mathbf{p}/m$
$\mathbf{v}_{particle} = \mathbf{p}/m$	$\mathbf{v}_{particle} = \mathbf{p}/m^+$

4 Nonrelativistic limit of the modified Klein-Gordon and Dirac equations

We now consider the nonrelativistic limit of modified Klein-Gordon and Dirac equations. The nonrelativistic limit of the ordinary Klein-Gordon and Dirac equations can be found in many relativistic quantum mechanics and quantum field theory books [5].

4.1- Ordinary Klein-Gordon Equation: Ordinary Klein-Gordon equation reads

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 c^2\right) \Psi(\vec{x}, t) = 0.$$

We define new operator $M = \sqrt{m^2 c^4 - c^2 \vec{\nabla}^2}$ and rewrite the Klein-Gordon equation as

$$-\frac{\partial^2}{\partial t^2} \Psi = M^2 \Psi.$$

By introducing new fields

$$\phi^\pm = \Psi \pm iM^{-1} \frac{\partial \Psi}{\partial t},$$

we can convert the Klein-Gordon equation to the first order equations

$$i \frac{\partial \phi^\pm}{\partial t} = \pm M \phi^\pm.$$

In the non-relativistic limit $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ or large c we have

$$M \simeq mc^2 - \frac{1}{2m} \vec{\nabla}^2.$$

For subtracting the rest mass energy, we transform to the fields $\tilde{\phi}^\pm = \exp(\pm imc^2 t) \phi^\pm$ and we reach to the Schrödinger equations

$$i \frac{\partial \tilde{\phi}^\pm}{\partial t} = \mp \frac{1}{2m} \vec{\nabla}^2 \tilde{\phi}^\pm.$$

The positive sign in the $\tilde{\phi}^\pm$ corresponds to the particle solution and the negative sign to the antiparticle case [6].

4.2- Modified Klein-Gordon Equation: Modified Klein-Gordon equation Eq.(3) in $O(E^3/k^3)$ approximation is

$$-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi + \frac{i}{k} \frac{\partial}{\partial t} \vec{\nabla}^2 \Phi - (m^2 c^2 - \vec{\nabla}^2) \Phi = 0.$$

This equation can be rewritten as

$$-\frac{\partial^2}{\partial t^2} \Phi = \left[m^2 c^4 - \left(1 + \frac{i}{k} \frac{\partial}{\partial t} \right) \vec{\nabla}^2 \right] \Phi.$$

By defining new operator

$$\tilde{M} = \sqrt{m^2 c^4 - c^2 \left(1 + \frac{i}{k} \frac{\partial}{\partial t} \right) \vec{\nabla}^2},$$

and introducing new fields

$$\psi^\pm = \Phi \pm i \tilde{M}^{-1} \frac{\partial \Phi}{\partial t}.$$

We can convert the modified Klein-Gordon equation to the first order equations

$$i \frac{\partial \psi^\pm}{\partial t} = \pm \tilde{M} \psi^\pm.$$

In the non-relativistic limit $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ or large c we have

$$\tilde{M} \simeq mc^2 - \frac{1}{2m} \left(1 + \frac{i}{k} \frac{\partial}{\partial t} \right) \vec{\nabla}^2.$$

By transforming to the fields $\tilde{\psi}^{\pm} = e^{\pm imc^2 t} \psi^{\pm}$, we reach to the Schrödinger equations

$$i \frac{\partial \tilde{\psi}^{\pm}}{\partial t} = \mp \frac{1}{2m^{\pm}} \vec{\nabla}^2 \tilde{\psi}^{\pm}.$$

As expected, the particle and the antiparticle states satisfy in the Schrödinger equations with the modified inertial masses m^+ and m^- .

4.3- Dirac Equation: Ordinary Dirac Equation is

$$(i\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} - m)\psi = 0.$$

Defining two-component form as

$$\psi(\vec{x}, t) = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

and using standard form of γ matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

we reach to coupled equations

$$\begin{cases} (E - m) \chi = \vec{\sigma} \cdot \vec{p} \eta \\ (E + m) \eta = \vec{\sigma} \cdot \vec{p} \chi \end{cases} \quad (14)$$

Note that $p^{\mu} = (E, \mathbf{p})$, $p_{\mu} = (E, -\mathbf{p})$ and $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

In the nonrelativistic limit for the particle solution, we take $E \simeq m$ and $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ which leads to $\eta \simeq (\vec{\sigma} \cdot \vec{p} / 2m) \chi$. Substituting this in Eq.(14) and introducing $\chi' = e^{+imt} \chi$, we reach to the Schrödinger equation

$$i \frac{\partial \chi'}{\partial t} \simeq -\frac{1}{2m} \vec{\nabla}^2 \chi'. \quad (15)$$

For the antiparticle solution in the nonrelativistic limit we take $E \simeq -m$. Doing as in particle case and introducing $\eta' = e^{-imt} \eta$, we can show that also η' satisfies in a similar Schrödinger equation

$$i \frac{\partial \eta'}{\partial t} \simeq +\frac{1}{2m} \vec{\nabla}^2 \eta'. \quad (16)$$

4.4- Modified Dirac Equation: Modified dirac equation Eq.(4) in the $O(E^2/k^2)$ approximation is

$$\left[i\gamma^0 \frac{1}{c} \frac{\partial}{\partial t} + i\gamma^i \frac{\partial}{\partial x^i} \left(1 + \frac{i}{2k} \frac{\partial}{\partial t} \right) - m \right] \tilde{\psi} = 0.$$

In this case, we obtain the two-component by introducing

$$\tilde{\psi}(\vec{x}, t) = \begin{pmatrix} \tilde{\chi} \\ \tilde{\eta} \end{pmatrix}$$

Doing as in the unmodified case, we reach to the following coupled equations

$$\begin{cases} [E - m] \tilde{\chi} = (1 + \frac{E}{2k}) \vec{\sigma} \cdot \vec{p} \tilde{\eta} \\ [E + m] \tilde{\eta} = (1 + \frac{E}{2k}) \vec{\sigma} \cdot \vec{p} \tilde{\chi} \end{cases}$$

In the nonrelativistic limit $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$, we take $E \simeq m$ for the particle solution and we have

$$\tilde{\eta} \simeq (1 + \frac{E}{2k}) \frac{\vec{\sigma} \cdot \vec{p}}{2m} \tilde{\chi}.$$

By introducing $\tilde{\chi}' = \exp(+imc^2 t) \tilde{\chi}$, we reach to the following Schrödinger equation

$$i \frac{\partial \tilde{\chi}'}{\partial t} \simeq -\frac{1}{2m^+} \vec{\nabla}^2 \tilde{\chi}'. \quad (17)$$

We note that $\tilde{\chi}'$ satisfies in the Schrödinger equation with the modified inertial mass m^+ .

Also, for the antiparticle we take $E \simeq -m$ and we have

$$\tilde{\chi} \simeq -(1 + \frac{E}{2k}) \frac{\vec{\sigma} \cdot \vec{p}}{2m} \tilde{\eta}.$$

If we define $\tilde{\eta}' = \exp(-imc^2 t) \tilde{\eta}$ we can show that $\tilde{\eta}'$ satisfies in a similar Schrödinger equation

$$i \frac{\partial \tilde{\eta}'}{\partial t} \simeq +\frac{1}{2m^-} \vec{\nabla}^2 \tilde{\eta}', \quad (18)$$

in which the modified mass m^- is given by Eq.(10)

5 Conclusion and some remarks

In this paper, we showed that the nonrelativistic limit of Amelino-Camelia DSR leads to the particle and the antiparticle with different inertial masses m^+ and m^- but the same rest mass. We interpreted this difference as a sign of the CPT violation.

Also, we studied the corresponding modified Klein-Gordon and Dirac equations, which reproduce the Schrödinger equations in the nonrelativistic limit with these modified masses for the particle and the antiparticle.

The ratio $|m^+ - m^-|/m$ is equal to $2mc^2/k$. We can use this ratio to find a lower bound on the amount of k . In DSR theories k is a fundamental constant with dimension of energy and in principle can be different from the Planck energy $E_p = \sqrt{\hbar c^5/G} \simeq 10^{19}$.

We compared the nonrelativistic limit of Amelino-Camelia and Magueijo-Smolon DSR. In this limit we reached to the $E = mc^2 + \mathbf{p}^2/2m^+$ for Amelino-Camelia DSR, also we $E = m^+c^2 + \mathbf{p}^2/2m$ for MS DSR in this limit [6]. These are seem to be the natural results of the dispersion relations. If we put the $O(E^3/k^3)$ approximation of Amelino-Camelia dispersion relation Eq.(1) and the MS dispersion relation [1] together:

$$\begin{cases} E^2 = \mathbf{p}^2c^2 + m^2c^4(1 - E/k)^2 & \text{Magueijo-Smolon} \\ E^2 \simeq \mathbf{p}^2c^2(1 + E/k) + m^2c^4 & \text{Amelino-Camelia} \end{cases}$$

Then the observation of their similarities and differences is easy, and the interpretation of the nonrelativistic limit of MS DSR as modifying rest mass and Amelino-Camelia DSR as modifying momentum is not difficult. Thus, in this limit the MS and Amelino-Camelia DRS theories seems to be complementary to each other.

6 Acknowledgment

We want to thanks to Dr. Ahmad Shariati for his kindly and useful discussions.

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This paper is devoted to the study of the nonrelativistic limit of Amelino-Camelia Doubly Special Relativity(DSR) and the corresponding modified Klein-Gordon and Dirac equations. These equations will be reduced to the Schrödinger equations for the particle and the antiparticle with different inertial masses. Amelino-Camelia and Magueijo-Smolín DSR are two main models of Doubly Special Relativity theories. M. Corradu and S. Mignemi recently have studied the non relativistic limit of the Magueijo-Smolín model. Their result is compared with our proposal, which shows that these two models are reciprocal to each other in the nonrelativistic limit. We also show that the different inertial masses leads to the CPT violation

Keywords: Doubly Special Relativity, Non-relativistic limit, Relativistic Quantum Mechanics.

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1 Introduction

Doubly Special Relativity (DSR) theories have been proposed ten years ago for the nonlinear modification of Special Relativity. These theories have two invariant scales, speed of the light c and the Planck energy $E_p = \sqrt{\hbar c^5/G} \simeq 10^{19}$ GeV. Magueijo-Smolín (MS) DSR [1] and Amelino-Camelia DSR [2, 3] are the two main models of DSR theories. Although, these model have similar structure, but they belong to different realizations of kappa-Poincaré algebras [4].

According to the special relativity and relativistic quantum mechanics, the nonrelativistic limit of the Klein-Gordon and Dirac equations is the Schrödinger equation. Also the particle and the antiparticle masses are the same, which can be seen easily from the dispersion relation $E = \pm\sqrt{m^2 + p^2}$.

Recently, M. Coraddu and S. Mignemi have studied the nonrelativistic limit of the MS DSR and the corresponding modified Klein-Gordon and Dirac equations [6]. They showed that the particle and the antiparticle rest masses are different, but their inertial masses are the same. Besides, they have obtained the nonrelativistic quantum mechanics from the modified Klein-Gordon and Dirac equations.

In the Doubly Special Relativity theories dispersion relation has been modified, which leads to the modified Klein-Gordon and Dirac equations. Thus, we expect different masses for the particle and the antiparticle, which also leads to the CPT violation. For examine these matters and investigating the nonrelativistic limit of the Klein-Gordon and Dirac equations. We like to study the nonrelativistic limit of Amelino-Camelia DSR and the corresponding modified Klein-Gordon and Dirac equations. Also, we compare the nonrelativistic limit of MS and Amelino-Camelia DSR.

In section 2, we summarize some basics of Amelino-Camelia DSR. In section 3, we study the non-relativistic limit of modified dispersion relation in Amelino-Camelia DSR, and make some comparisons with the nonrelativistic limit of MS DSR. The non-relativistic limits of ordinary and modified Klein-Gordon and Dirac equations is presented in section 4. Then, in section 5 we have some conclusions.

2 Amelino-Camelia DSR

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in which k is the Planck energy E_p and a is an index varies from 1 to 3. He proposed the modified boosts

$$B_a = ip_a \frac{\partial}{\partial E} + i \left(\frac{1}{2k} \mathbf{p}^2 + k \frac{1 - e^{-2E/k}}{2} \right) \frac{\partial}{\partial p_a} - i \frac{p_a}{k} \left(p_b \frac{\partial}{\partial p_b} \right), \quad (2)$$

which leave invariant the mentioned dispersion relation. Also, for these modified boosts we can find new transformations, which will be different from the Lorentz transformations [7].

The modified Klein-Gordon equation is obtained from (1) by substituting differential operators for $E = i \frac{\partial}{\partial t}$ and $\mathbf{p} = -i \vec{\nabla}$ in the fashion standard in quantum mechanics

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3 Nonrelativistic limit

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In Amelino-Camelia DSR, we expand modified dispersion relation (1) in $O(E^3/k^3)$ approximation which becomes

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Solving this equation as a second order equation for E leads to

$$E = \frac{\mathbf{p}^2 c^2 / k \pm \sqrt{\mathbf{p}^4 c^4 / k^2 + 4(\mathbf{p}^2 c^2 + m^2 c^4)}}{2}.$$

In the $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ limit, we have

$$E \simeq \frac{\mathbf{p}^2 c^2}{2k} \pm mc^2 \left(1 + \frac{\mathbf{p}^2}{2m^2 c^2}\right).$$

The positive sign corresponds to the classical non-relativistic limit

$$E = mc^2 + \frac{\mathbf{p}^2}{2m^+}, \quad (8)$$

in which we assume

$$m^+ = \frac{m}{1 + \frac{mc^2}{k}}. \quad (9)$$

Thus, the particle inertial mass has been modified by amount of mc^2/k , but the rest mass of particle remains unmodified. By inertial mass we mean the mass which appears in dominator of $\mathbf{p}^2/2m^+$. Also, by defining

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we can interpret the negative sign solution as an antiparticle

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Different inertial mass m^+ and m^- in the equations (9) and (10) leads to the violation of CPT invariance. However, the interpretation of the inertial mass as the mass of an elementary particle may have some difficulties. In fact dispersion relation Eq.(7) is not invariant under space inversion and time reversal which imply the violation of CPT invariance.

We can compare this case with the nonrelativistic limit of the MS model [6]. They reached to the modified relation ($E = m^+c^2 + \frac{\mathbf{p}^2}{2m}$) instead of Eq.(8) in which m^+ has the same value with Eq.(9). In their proposal the rest mass of the particle is modified, and the inertial mass remains the same. Thus, the nonrelativistic limit of Magueijo-Smolín DSR is reciprocal to the nonrelativistic limit of Amelino-Camelia DSR.

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We summarize comparison of the nonrelativistic limit of Amelino-Camelia and MS doubly special relativity in the following table.

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$\mathbf{v}_g = \mathbf{p}/m^+$	$\mathbf{v}_g = \mathbf{p}/m$
$\mathbf{v}_{particle} = \mathbf{p}/m$	$\mathbf{v}_{particle} = \mathbf{p}/m^+$

4 Nonrelativistic limit of the modified Klein-Gordon and Dirac equations

We now consider the nonrelativistic limit of modified Klein-Gordon and Dirac equations. The nonrelativistic limit of the ordinary Klein-Gordon and Dirac equations can be found in many relativistic quantum mechanics and quantum field theory books [5].

4.1- Ordinary Klein-Gordon Equation: Ordinary Klein-Gordon equation reads

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 c^2\right) \Psi(\vec{x}, t) = 0.$$

We define new operator $M = \sqrt{m^2 c^4 - c^2 \vec{\nabla}^2}$ and rewrite the Klein-Gordon equation as

$$-\frac{\partial^2}{\partial t^2} \Psi = M^2 \Psi.$$

By introducing new fields

$$\phi^\pm = \Psi \pm iM^{-1} \frac{\partial \Psi}{\partial t},$$

we can convert the Klein-Gordon equation to the first order equations

$$i \frac{\partial \phi^\pm}{\partial t} = \pm M \phi^\pm.$$

In the non-relativistic limit $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ or large c ,

$$M \simeq mc^2 - \frac{1}{2m} \vec{\nabla}^2.$$

For subtracting the rest mass energy, we transform to the fields $\tilde{\phi}^\pm = \exp(\pm imc^2 t) \phi^\pm$ and which leads to the Schrödinger equations

$$i \frac{\partial \tilde{\phi}^\pm}{\partial t} = \mp \frac{1}{2m} \vec{\nabla}^2 \tilde{\phi}^\pm.$$

The positive sign in the $\tilde{\phi}^\pm$ corresponds to the particle solution and the negative sign to the antiparticle case [6].

4.2- Modified Klein-Gordon Equation: Modified Klein-Gordon equation Eq.(3) in $O(E^3/k^3)$ approximation is

$$-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi + \frac{i}{k} \frac{\partial}{\partial t} \vec{\nabla}^2 \Phi - (m^2 c^2 - \vec{\nabla}^2) \Phi = 0.$$

This equation can be rewritten as

$$-\frac{\partial^2}{\partial t^2} \Phi = \left[m^2 c^4 - \left(1 + \frac{i}{k} \frac{\partial}{\partial t} \right) \vec{\nabla}^2 \right] \Phi.$$

By defining new operator

$$\tilde{M} = \sqrt{m^2 c^4 - c^2 \left(1 + \frac{i}{k} \frac{\partial}{\partial t} \right) \vec{\nabla}^2},$$

and introducing new fields

$$\psi^\pm = \Phi \pm i \tilde{M}^{-1} \frac{\partial \Phi}{\partial t}.$$

We can convert the modified Klein-Gordon equation to the first order equations

$$i \frac{\partial \psi^\pm}{\partial t} = \pm \tilde{M} \psi^\pm.$$

In the non-relativistic limit $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ or large c ,

$$\tilde{M} \simeq m c^2 - \frac{1}{2m} \left(1 + \frac{i}{k} \frac{\partial}{\partial t} \right) \vec{\nabla}^2.$$

By transforming to the fields $\tilde{\psi}^\pm = e^{\pm i m c^2 t} \psi^\pm$, we reach to the Schrödinger equations

$$i \frac{\partial \tilde{\psi}^\pm}{\partial t} = \mp \frac{1}{2m^\pm} \vec{\nabla}^2 \tilde{\psi}^\pm.$$

As expected, the particle and the antiparticle states satisfy in the Schrödinger equations with the modified inertial masses m^+ and m^- .

4.3- Dirac Equation: Ordinary Dirac Equation is

$$(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m) \psi = 0.$$

Defining two-component form as

$$\psi(\vec{x}, t) = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

and using the standard form of γ matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

we reach to coupled equations

$$\begin{cases} (E - m) \chi = \vec{\sigma} \cdot \vec{p} \eta \\ (E + m) \eta = \vec{\sigma} \cdot \vec{p} \chi \end{cases} \quad (14)$$

Note that $p^\mu = (E, \mathbf{p})$, $p_\mu = (E, -\mathbf{p})$ and $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

In the nonrelativistic limit for the particle solution, we take $E \simeq m$ and $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$ which leads to $\eta \simeq (\vec{\sigma} \cdot \vec{p} / 2m) \chi$. Substituting this in Eq.(14) and introducing $\chi' = e^{+imt} \chi$, we reach to the Schrödinger equation

$$i \frac{\partial \chi'}{\partial t} \simeq -\frac{1}{2m} \vec{\nabla}^2 \chi'. \quad (15)$$

For the antiparticle solution in the nonrelativistic limit we take $E \simeq -m$. Doing as in particle case and introducing $\eta' = e^{-imt} \eta$, we reach to a similar Schrödinger equation

$$i \frac{\partial \eta'}{\partial t} \simeq +\frac{1}{2m} \vec{\nabla}^2 \eta'. \quad (16)$$

4.4- Modified Dirac Equation: Modified dirac equation Eq.(4) in the $O(E^2/k^2)$ approximation is

$$\left[i\gamma^0 \frac{1}{c} \frac{\partial}{\partial t} + i\gamma^i \frac{\partial}{\partial x^i} \left(1 + \frac{i}{2k} \frac{\partial}{\partial t} \right) - m \right] \tilde{\psi} = 0.$$

In this case, we obtain the two-component by introducing

$$\tilde{\psi}(\vec{x}, t) = \begin{pmatrix} \tilde{\chi} \\ \tilde{\eta} \end{pmatrix}$$

Doing as in the unmodified case, we reach to the following coupled equations

$$\begin{cases} [E - m] \tilde{\chi} = (1 + \frac{E}{2k}) \vec{\sigma} \cdot \vec{p} \tilde{\eta} \\ [E + m] \tilde{\eta} = (1 + \frac{E}{2k}) \vec{\sigma} \cdot \vec{p} \tilde{\chi} \end{cases}$$

In the nonrelativistic limit $(\mathbf{p}^2 c^2 / m^2 c^4) \ll 1$, we take $E \simeq m$ for the particle solution which leads to

$$\tilde{\eta} \simeq (1 + \frac{E}{2k}) \frac{\vec{\sigma} \cdot \vec{p}}{2m} \tilde{\chi}.$$

By introducing $\tilde{\chi}' = \exp(+imc^2 t) \tilde{\chi}$, we obtain the following Schrödinger equation

$$i \frac{\partial \tilde{\chi}'}{\partial t} \simeq -\frac{1}{2m^+} \vec{\nabla}^2 \tilde{\chi}'. \quad (17)$$

Note that $\tilde{\chi}'$ satisfies in the Schrödinger equation with the modified inertial mass m^+ .

Also, for the antiparticle we have $E \simeq -m$ and $\tilde{\chi} \ll \tilde{\eta}$. If we define $\tilde{\eta}' = \exp(-imc^2 t) \tilde{\eta}$ we can show that $\tilde{\eta}'$ satisfies in a similar Schrödinger equation

$$i \frac{\partial \tilde{\eta}'}{\partial t} \simeq +\frac{1}{2m^-} \vec{\nabla}^2 \tilde{\eta}', \quad (18)$$

in which the modified mass m^- is given by Eq.(10)

5 Conclusion and some remarks

In this paper, we showed that the nonrelativistic limit of Amelino-Camelia DSR leads to the particle and the antiparticle with different inertial masses m^+ and m^- . We interpreted this difference as a sign of the CPT violation. Also, we studied the corresponding modified Klein-Gordon and Dirac equations, which reduce to the Schrödinger equations in the nonrelativistic limit.

In DSR theories k is a fundamental constant that we put it equal to the Planck energy $E_p = \sqrt{\hbar c^5 / G} \simeq 10^{19}$. However, k can access any different values. Generally we assume $k = \alpha E_p$ in which $0 \leq \alpha \leq 2$. We found $2mc^2/k$ for $|m^+ - m^-|/m$ i.e. the relative difference between the particle and the antiparticle masses. We can use this ratio to find a lower bound on the amount of k .

As mentioned we had $E = mc^2 + \mathbf{p}^2/2m^+$ and $E = m^+c^2 + \mathbf{p}^2/2m$ for the nonrelativistic limit of Amelino-Camelia and MS dispersion relations. This reciprocity is the natural consequences of DSR theories. If we put the Amelino-Camelia and MS dispersion relations together:

$$\left\{ \begin{array}{ll} E^2 = \mathbf{p}^2c^2 + m^2c^4(1 - E/k)^2 & \text{Magueijo-Smolín} \\ E^2 \simeq \mathbf{p}^2c^2(1 + E/k) + m^2c^4 & \text{Amelino-Camelia} \end{array} \right.$$

Then the interpretation of the nonrelativistic limit of MS DSR as modifying rest mass and Amelino-Camelia DSR as modifying momentum is straightforward. Thus, in this limit MS and Amelino-Camelia DRS theories are complementary to each other.

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